Tutorial 4 MS 414/MD308/MI 308

(1) Prove that $X(t)$ is a solution to $X' = A(t)X,\ X(t_0) = X_0$ on I if and only if

$$X(t) = X_0 + \int_{t_0}^{t} A(s)X(s)ds,\ t \in I.$$

(2) Show that the column of the matrix $\phi(t) = \begin{pmatrix} 1 & t & t^2 \\ 0 & 2 & t \\ 0 & 0 & 0 \end{pmatrix}$ are linearly independent but $\phi(t) = 0$. Does it satisfy $X' = A(t)X$?

(3) Let ϕ be a fundamental matrix for the system $X' = AX$. Find the fundamental matrix for the system $X' = -A^T X$

(4) Show that $C\phi(t)$, where C is a constant matrix and ϕ is a fundamental matrix of $X' = A(t)X$, need not be a solution matrix of $X' = A(t)X$.

(5) Find the fundamental matrix for the $X' = AX$ where

(a) $\begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 & 8 \\ -2 & 2 & 1 \\ -3 & 0 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 3 & 4 \\ 1 & 5 & -1 \\ 0 & 2 & 0 \end{pmatrix}$

(6) Show that $\phi(t) = \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}$ is a fundamental matrix for the system $X' = A(t)X$, where $A(t) = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix}$ for $t \neq 0$. Find the solution ϕ to $X' = A(t)X + b(t)$, where $b(t) = \begin{pmatrix} t^4 \\ t^2 \end{pmatrix}$ and $\phi(1) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

(7) Consider the following ODE

$$x'' + p(t)x' + q(t)x = f(t)$$

with p, q, f continuous on some interval I. Let ϕ_1 and ϕ_2 be two linearly independent solution of the hoomgeneous equation associated with the given equation. Write the given equation in matrix form and show that $\Phi(t) = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{pmatrix}$ is a fundamental matrix of the associated homogeneous system on I.

(8) For a nonsingular $n \times n$ matrix T show that $T^{-1}(\exp A)T = \exp(T^{-1}AT)$.

(9) Obtain the phase portrait of the system $X' = AX$, where

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

(b) $x'' + 4x = 0$ (e) $x'' + 3x' + x = 0$

(10) Determine the nature of the critical points in the following systems and sketch the phase portrait

(a) $x_1' = x_1,\ x_2' = x_2$
(b) $x_1' = -2x_1 - x_2,\ x_2' = -5x_1 - 6x_2$
(c) $x_1' = 4x_1 + x_2,\ x_2' = 3x_1 + 6x_2$
(d) $x_1' = 5x_1 + 3x_2,\ x_2' = -3x_1 - x_2$
(e) $x_1' = 2x_1 - 3x_2,\ x_2' = x_1 - 2x_2$

(11) Find a suitable Liapunov function for the following problem . Also discuss the stability.

(a) $x_1' = -x_1^3 + x_1x_2^2,\ x_2' = -2x_1^2x_2 - x_2^3$
(b) $x_1' = x_1^3 - x_2^3,\ x_2' = 2x_1x_2^2 + 4x_1^2x_2 + 2x_2^3$
(c) $x_1' = -x_1^3 + 2x_2^3,\ x_2' = -2x_1x_2^2$
(d) $x_1' = -\frac{1}{2}x_1^3 + 2x_1x_2^2,\ x_2' = -x_2^3$